Math 254A Lecture 3 Notes

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1 Turning Set Functions Into Point Functions

1.1 Recap + dealing with the empty set

Last time, we had a finite alphabet A, and given $U \subseteq P(A)$, we looked at $T_n(U) = \{x \in A^n : p_x \in U\}$. We looked at the asymptotic behavior of the size of this set without relying on explicit formulae. We defined $S_n(U) = \log |T_n(U)|$.

What if $T_n(U) = \varnothing$? Here are two answers.

1. If $U \neq \varnothing$ is open, if we pick $p \in U$, let n be very large and pick $X \sim p^{\times n}$. Then $\mathbb{P}(p_X \in U) \to 1$ as $n \to \infty$ by the Weak Law of Large Numbers. So $T_n(U) \neq \varnothing$ for all sufficiently large n.

2. We should let S_n take the value $-\infty$. This will be fine, as long as we're not subtracting negative infinities or multiplying. This is the better answer

Last time, we showed that $S_n(U)$ is superadditive if U is convex:

$$S_{n+m}(U) \geq S_n(U) + S_m(U).$$

By Fekete's lemma, $S(U) = \lim_n \frac{1}{n} S_n(U)$ exists and equals $\sup_n \frac{1}{n} S_n(U)$. This tells us that

$$|T_n(U)| = e^{S(U)n + o(n)}.$$

This produces a set function $S : \{\text{convex open subsets of } P(A)\} \to [-\infty, \infty]$. We would like a point function $S : P(A) \to [-\infty, \infty]$ such that $s(U) = \sup\{S(p) : p \in U\}$.

1.2 General considerations: when do set functions give rise to point functions?

We will step away for a while to a more abstract setting: Let X be a topological space, let $\mathcal{U}$ be a cover of X by open sets, and let $S : \mathcal{U} \to [-\infty, \infty]$. When is there a point function $S : X \to [-\infty, \infty]$ such that $S(U) = \sup\{S(x) : x \in U\}$?

The first necessary condition is

(S1) If $U, U_1, \ldots, U_k \in \mathcal{U}$ and $U \subseteq U_1 \cup \cdots \cup U_k$, then $S(U) \leq \max_i S(U_i)$.

Unfortunately, this condition is not sufficient, but we will give a sufficient condition later.

Aside: Call S **locally finite** if for every $x \in X$, there is some $U \in \mathcal{U}$ such that $x \in U$ and $S(U) < \infty$.

Now let's define $S(x) := \inf\{S(U) : U \in \mathcal{U}, U \ni x\}$. Then the following is true.

Lemma 1.1.

$$S(U) \geq \sup\{S(x) : x \in U\}.$$

Lemma 1.2. *The point function S must be upper semicontinuous.*

Proof. If $S(x) < a$, then there exists some $U \in \mathcal{U}$ with $x \in U$ and $S(U) < a$, but then $U \subseteq \{S < a\}$. $\square$

Now suppose that $K \subseteq X$ is compact. We want to define S for these types of sets, rather than just open sets. Define

$$S(K) := \inf\{\max_i S(U_i) : U_1, \ldots, U_k \in \mathcal{U}, K \subseteq U_1 \cup \cdots \cup U_k\}.$$

Remark 1.1. If S is locally finite, then $S(K) < \infty$ for all compact K.

Remark 1.2. If $K = \{x\}$, then $S(K) = S(x)$.

Lemma 1.3. *If $U \in \mathcal{U}$ and $\overline{U}$ is compact, then $S(U) \leq S(\overline{U})$.*

This is the first moment where we actually use the property (S1).

Proof. If $U_1 \ldots, U_k \supseteq \overline{U} \supseteq U$, then by (S1), $S(U) \leq \max_i S(U_i)$. $\square$

Corollary 1.1. *If $U \in \mathcal{U}$ is also compact, then $S(U)$ is unambiguous.*

Proof. The previous lemma gives $S(U) \leq S(\overline{U}) \leq S(U)$. $\square$

Lemma 1.4. *For every compact $K \neq \varnothing$, we have*

$$S(K) = \sup\{S(x) : x \in K\}.$$

Proof. If $\{x\} \subseteq K$, then

$$S(x) = S(\{x\}) \leq S(K).$$

For the other direction, if $\sup_{x \in K} s(x) = \infty$, we are done. So assume that this is $< \infty$ and let $a > \sup_K S(x)$. Then for any $x \in K$, there is some $V_n \in \mathcal{U}$ with $S(V_n) < a$. K is compact, so there exist $x_1, \ldots, x_k$ with $K \supseteq V_{x_1} \cup \cdots \cup V_{x_k}$, and so

$$S(K) \leq \max_i S(V_{x_i}) < a.$$

Taking the inf over as gives

$$S(K) \leq \sup_K S(x). \quad \square$$

Here is the second necessary condition on the set function S:

(S2) ("Inner regularity") $S(U) = \sup\{S(K) : K \text{ is compact}, K \subseteq U\}$

Lemma 1.5. *If (S1) and (S2) hold, then* $S(U) = \sup\{S(x) : x \in U\}$.

Proof. We already know $\geq$. For the reverse inequality, use (S2): It is enough to show that

$$\sup_K S(x) = S(K) \leq \sup_U S(x).$$

for all compact $K \subseteq U$. This inequality holds by the previous lemma. □

1.3 The settings we will apply this general theory to

The main settings we care about are:

1. Z is some "nice" topological space (usually a compact metric space), $X = M(Z)$, the finite signed Borel measures on Z, and $\mathcal{U}$ is the collection of convex subsets open for the weak topology defined by $C_b(Z)$ (i.e. the weak* topology if Z is a compact metric space).

2. $X = \mathbb{R}^d$ and $\mathcal{U}$ is the collection of convex open sets.

A suitable intermediate generality for us to cover these two cases will be: X is a locally convex topological vector space and $\mathcal{U}$ is the collection of open convex subsets.

Next time, we will

- find conditions making the point function S concave,
- observe a general "sequence counting" situation where those conditions hold.